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The Theory of J-Selfadjoint Extensions of J-Symmetric Operators

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1. INTRODUCTION

The study of boundary value problems involving linear differential equations with complex-valued coefficients is becoming a well-established area of analysis. The motivation and background for considering the relevant differential expressions, together with the development of the associated theory may be found in [10–13, 17–22], together with the references given therein. Such expressions are not formally symmetric and hence the spectral theory of selfadjoint operators is not applicable.

To study such problems, Glazman introduced the concept of a J-symmetric operator in [5]: In a complex Hilbert space $\mathscr{H}$, let J be a conjugation operator on $\mathscr{H}$, i.e., J is a conjugate-linear involution with $(Jx, Jy) = (y, x)$ for all x and y in $\mathscr{H}$. A linear operator A in $\mathscr{H}$ is said to be *J-symmetric* if its domain, $\mathscr{D}(A)$, is dense in $\mathscr{H}$ and A satisfies

$$A \subseteq JA^*J \tag{1.1}$$

in the usual sense of operator inclusion, where A^* is the (Hilbert space) adjoint of A. This last condition is equivalent to requiring

$$(Jx, Ay) = (JAx, y) \tag{1.2}$$

for all x and y in $\mathscr{D}(A)$. If, further,

$$A = JA^*J, \tag{1.3}$$

then A is said to be *J-selfadjoint*.

It was seen in [10, 26] that certain nonsymmetric differential expressions generate J-symmetric operators in the Hilbert space $L^2(a, b)$, where J is the

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usual operation of complex conjugation of functions in $L^2(a, b)$. Whereas in the case of symmetric differential expressions one is concerned with the boundary conditions which determine associated selfadjoint operators, one is now concerned with those which determine J-selfadjoint operators. A partial characterization of the appropriate boundary conditions was derived by Zhikhar [26] but a more general and more useful theory was developed by Knowles [10]. Knowles gave a complete solution to the problem of describing all the J-selfadjoint extensions of any given J-symmetric operator A provided its regularity field (see Sect. 2), $\Pi(A)$, is nonempty. This was then applied to describing explicitly the boundary conditions which determine the J-selfadjoint operators associated with a general, formally J-symmetric linear differential expression of even order with complex-valued coefficients.

Given such an expression, it is in practice difficult however, to determine whether the appropriate J-symmetric operator has empty or nonempty regularity field. The main criteria known are derived or referred to in [18, 20]. We note that the restriction on the regularity field occurs not only in [10, 26] but also in [12, 13, 21] and related papers.

It is the main purpose, therefore, in the present work, to remove the restriction that the regularity field be nonempty, from as much as possible of the theory of J-selfadjoint extensions of J-symmetric operators. This is achieved in Section 3. In Section 4 the general theory is applied to the appropriate differential expressions. The results concerning differential operators generalize all of those given in [10, Sect. 4] by removing the condition on the regularity field, their statement being otherwise identical. Also, the methods of proof employed here are both simpler and shorter than those of [10].

Last in Section 5 the case when the regularity field of the differential operator is empty is considered and this is illustrated with an example of McLeod [14]. The present work answers some previously open questions concerning this particular example, (cf. Remark 2 following Theorem 3.6 of [10]). The details of the analogue of the Weyl limit-point–limit-circle dichotomy are also given in this last section.

2. Preliminaries

In this section we give some of the definitions and results which will be needed later.

Definition 2.1. Let A be a closed, linear operator with domain $\mathscr{D}(A)$ dense in a complex Hilbert space $\mathscr{H}$. The *regularity field* of A, $\Pi(A)$, is defined to be the set of all complex numbers λ for which $\|(A - \lambda I)x\| >$

$k_\lambda \|x\|$ for some positive number k_λ depending only on λ, and all x in $\mathscr{D}(A)$; here, I denotes the identity operator on $\mathscr{H}$.

The results of [10,26] were based on the following decomposition of domains of operators, where the symbol $\dot{+}$ denotes a direct sum which need not be an orthogonal direct sum.

LEMMA 2.2 [26, Theorem 3]. *Let A be a closed J-symmetric operator and suppose $\Pi(A)$ is not empty. For an arbitrary, fixed $\lambda_0 \in \Pi(A)$, let A' be any J-selfadjoint extension of A for which $\lambda_0 \in \Pi(A')$. Then*

$$\mathscr{D}(JA^*J) = \mathscr{D}(A) \dot{+} (A' - \lambda_0 I)^{-1} \mathscr{N}_{\lambda_0} \dot{+} J\mathscr{N}_{\lambda_0},$$

where

$$\mathscr{N}_{\lambda_0} = \{x \in \mathscr{D}(A^*) : (A^* - \bar{\lambda}_0 I)\, x = 0\}.$$

The existence of such an operator A' is guaranteed by [26, Theorem 2]. Denoting the dimension of the subspace $\mathscr{N}_{\lambda_0}$ by $m(\lambda_0)$, clearly $0 \leqslant m(\lambda_0) \leqslant \infty$. Since $(A' - \lambda_0 I)^{-1}$ is one-to-one, $2m(\lambda_0)$ is equal to the dimension of $\mathscr{D}(JA^*J)$ modulo $\mathscr{D}(A)$, i.e., to the dimension of the quotient space $\mathscr{D}(JA^*J)/\mathscr{D}(A)$, which is independent of λ_0 in $\Pi(A)$. Thus $m(\lambda_0)$ has a constant value over $\Pi(A)$. This justifies being able to make the following definition.

DEFINITION 2.3 [26]. Let A be a closed J-symmetric operator with $\Pi(A)$ not empty. For any λ_0 in $\Pi(A)$ we define the *defect number* of A, written def A, to be the dimension of the subspace $\mathscr{N}_{\lambda_0}$.

The following is then an immediate consequence of Lemma 2.2 and Definition 2.3.

LEMMA 2.4. *If A is a closed J-symmetric operator with $\Pi(A)$ not empty, then*

$$\dim \mathscr{D}(JA^*J)/\mathscr{D}(A) = 2.\mathrm{def}\, A.$$

Here and elsewhere, we use dim to denote the dimension of a space.

If however, we wish to cover the case when $\Pi(A)$ may be empty, it is clear that no decomposition of domains such as that given in Lemma 2.2 is obtainable. Likewise, we shall need to replace Definition 2.3 by a new definition of the defect number of A, which must not refer to a number λ_0 in $\Pi(A)$. This will be achieved in the next section. Meanwhile we now give the two main known results concerning J-symmetric operators, which permit the regularity field to be empty.

LEMMA 2.5 ([4], cf. [9]). *Every J-symmetric operator in $\mathscr{H}$ has a J-selfadjoint extension in $\mathscr{H}$.*

LEMMA 2.6 [9]. *If A is a J-symmetric operator and A' is a J-symmetric extension of A, then A' is J-selfadjoint if and only if it is maximal J-symmetric.*

By saying A is maximal J-symmetric, we mean it possesses no proper J-symmetric extensions. From now on, unless otherwise specified, we let A denote a J-symmetric operator in $\mathscr{H}$. We observe that if $\tilde{A}$ is a J-symmetric extension of A, then

$$A \subseteq \tilde{A} \subseteq J\tilde{A}^*J \subseteq JA^*J.$$

Consequently the most general J-symmetric extension of A must be a restriction of JA^*J to a subset of $\mathscr{D}(JA^*J)$ which is a linear manifold containing $\mathscr{D}(A)$. Since A has a closure $\bar{A}$, and since a J-selfadjoint operator is closed (by (1.3)), we need only search among the closed J-symmetric extensions of $\bar{A}$ to find all the J-selfadjoint extensions of A. Concerning notation, we use $\subseteq$ to denote operator inclusion and $\subset$ if the inclusion is strict, i.e., if the operators cannot be equal.

As in [10], we define an inner product on $\mathscr{D}(JA^*J)$ by

$$(x, y)^* = (Jx, Jy) + (A^*Jx, A^*Jy). \tag{2.1}$$

Since J is a conjugation this is equivalent to

$$(x, y)^* = (y, x) + (JA^*Jy, JA^*Jx). \tag{2.2}$$

With this inner product, $\mathscr{D}(JA^*J)$ becomes a Hilbert space (see [1, p. 1225]). Using $\ominus$ to denote the orthogonal complement with respect to this inner product it is clear that the quotient space $\mathscr{D}(JA^*J)/\mathscr{D}(A)$ is isomorphic to $\mathscr{D}(JA^*J) \ominus \mathscr{D}(A)$, i.e.,

$$\mathscr{D}(JA^*J)/\mathscr{D}(A) \cong \mathscr{D}(JA^*J) \ominus \mathscr{D}(A). \tag{2.3}$$

Likewise we shall use $\oplus$ to denote an orthogonal sum with respect to this inner product.

Finally, we note that $\mathbb{R}$, $\mathbb{C}$ are used to denote the sets of real and complex numbers respectively, and that $(x \in \mathscr{D})$ is used to mean that the aforegoing statement is to hold for all x in $\mathscr{D}$.

3. EXTENSIONS OF GENERAL J-SYMMETRIC OPERATORS

In this section we prove three simple results, the first of which enables us to generalize the definition of defect number.

THEOREM 3.1. *Let A be a closed J-symmetric operator and A' any J-selfadjoint extension of A, then*

$$\dim \mathscr{D}(JA^*J)/\mathscr{D}(A') = \dim \mathscr{D}(A')/\mathscr{D}(A).$$

Proof. If A is J-selfadjoint, then by Lemma 2.6, $A' = A = JA^*J$ and the result is trivial. So let us suppose $A \subset JA^*J$ and thus $\mathscr{D}(A) \subset \mathscr{D}(JA^*J)$. Let $m = \dim \mathscr{D}(A')/\mathscr{D}(A)$ and let $L(U)$ denote the linear span of the elements of a set U.

If $m < \infty$, then there are m elements of $\mathscr{D}(A')$, $\{y_i\}_{i=1}^m$, which are linearly independent modulo $\mathscr{D}(A)$ and such that

$$\mathscr{D}(A') = \mathscr{D}(A) \dotplus L(y_1,..., y_m). \tag{3.1}$$

For each i, $1 \leqslant i \leqslant m$, we define an operator A_i by

$$\begin{aligned} \mathscr{D}(A_i) &= \mathscr{D}(A) \dotplus L(y_1,..., y_i), \\ A_i y &= A'y \qquad \text{for } y \in \mathscr{D}(A_i). \end{aligned} \tag{3.2}$$

Since A is closed, so is A_i closed, $1 \leqslant i \leqslant m$. Likewise each A_i is a linear operator. Furthermore, it is clear that $A_i \subset A_{i+1}$ for each i, and hence $A_i^* \supseteq A_{i+1}^*$ for each i. If $A_i^* = A_{i+1}^*$, then $A_i^{**} = A_{i+1}^{**}$, which is the same as saying $A_i = A_{i+1}$, which does not occur, so in fact we have $A_i^* \supset A_{i+1}^*$. It then follows that

$$A' = JA'^*J = JA_m^*J \subset JA_{m-1}^*J \subset \cdots \subset JA_1^*J \subset JA^*J, \tag{3.3}$$

and hence since each inclusion is strict,

$$\begin{aligned} &\dim \mathscr{D}(JA^*J)/\mathscr{D}(A') \geqslant m \\ \text{i.e.,} \quad &\dim \mathscr{D}(JA^*J)/\mathscr{D}(A') \geqslant \dim \mathscr{D}(A')/\mathscr{D}(A). \end{aligned} \tag{3.4}$$

For the opposite inequality we begin by considering $\mathscr{D}(JA^*J)/\mathscr{D}(A')$ and use a similar argument. The only extra step needed is the fact that $JB^*J = (JBJ)^*$ for any densely defined operator B [17, Lemma 2.12]. This proves the theorem for finite m.

If $m = \infty$ then in (3.1) we have the linear span of an infinite set so we may construct an infinite sequence of operators of the form (3.2) which then satisfy strict inclusions like those in (3.3). The above argument then shows that if $\dim \mathscr{D}(A')/\mathscr{D}(A) = \infty$, then $\dim \mathscr{D}(JA^*J)/\mathscr{D}(A') = \infty$. The reverse implication may be proved in a similar manner. ∎

Remark 1. Lemma 2.5 guarantees the existence of at least one such operator A'.

Remark 2. The conclusion of Theorem 3.1 is that if one of the two dimensions is finite, so is the other, and they are equal, whereas if one is infinite so is the other. No distinction is being made between degrees of infinity.

Remark 3. Intuitively this theorem may be thought of, as saying that the domain of a J-selfadjoint extension of A is "midway" between the domains of A and JA^*J.

It follows from Theorem 3.1 that if A' is any J-selfadjoint extension of a closed J-symmetric operator A then

$$\dim \mathscr{D}(JA^*J)/\mathscr{D}(A) = 2.\dim \mathscr{D}(A')/\mathscr{D}(A). \tag{3.5}$$

Comparing (3.5) with Lemma 2.4 leads us to a natural extension of Definition 2.3 of the defect number of a J-symmetric operator.

DEFINITION 3.2. Let A be any closed J-symmetric operator. We define the (*generalized*) *defect number* of A, $\operatorname{def} A$, to be one half of the dimension of $\mathscr{D}(JA^*J)/\mathscr{D}(A)$.

Theorem 3.1 guarantees that $\operatorname{def} A$ is either a nonnegative integer or else infinite. Definition 3.2 is equivalent to $\operatorname{def} A = \dim \mathscr{D}(A')/\mathscr{D}(A)$ for any J-selfadjoint extension A' of A. It is an immediate consequence of this definition that

$$\dim \mathscr{D}(JA^*J)/\mathscr{D}(A) = 2.\operatorname{def} A.$$

Our definition of the generalized defect number holds for any J-symmetric operator. In [8], Kauffman defines the mean deficiency index for an arbitrary differential expression. It will be seen later that for the minimal operator generated by a differential expression which is formally J-symmetric, the two quantities coincide, i.e., in cases when they are both defined they are the same.

In practice it is likely to be easier to determine the value of the defect number from Definition 2.3 than from Definition 3.2. However, without the additional information that the regularity field is not empty it seems unlikely that anything more explicit than Definition 3.2 could be used for a generalized defect number.

One of the major results of [10, Sect. 3] concerns the determination of which J-symmetric extensions of any given closed J-symmetric operator with finite defect number are in fact J-selfadjoint. The methods employed there, using multivalued mappings would seem to be unnecessary, as the result we now prove suffices in place of [10, Theorem 3.8].

THEOREM 3.3. *Let A be any closed J-symmetric operator with* $\operatorname{def} A < \infty$ *and let A' be a J-symmetric extension of A, then A' is J-selfadjoint if and only if* $\dim \mathscr{D}(A')/\mathscr{D}(A) = \operatorname{def} A$.

Proof. Theorem 3.1 proves the result in one direction. For the reverse implication, suppose $\dim \mathscr{D}(A')/\mathscr{D}(A) = \operatorname{def}(A)$. By Lemma 2.5 A' has a J-selfadjoint extension A'', which by Theorem 3.1 satisfies $\dim \mathscr{D}(A'')/\mathscr{D}(A) = \operatorname{def} A$. It is therefore clear that $\dim \mathscr{D}(A'')/\mathscr{D}(A') = 0$, i.e., $A'' = A'$, and so A' is itself J-selfadjoint. ▮

Although we cannot hope for the type of decomposition given in Lemma 2.2 when $\Pi(A)$ may be empty, we do need some form of comparison between $\mathscr{D}(A)$ and $\mathscr{D}(JA^*J)$ if we are to be able to evaluate $\operatorname{def} A$. To this end we prove

THEOREM 3.4. *If A is a closed J-symmetric operator, then*

$$\mathscr{D}(JA^*J) = \mathscr{D}(A) \oplus \{y \in \mathscr{D}(A^*JA^*J) : A^*JA^*Jy = -y\}.$$

Proof. First, if $y \in \mathscr{D}(JA^*J) \ominus \mathscr{D}(A)$, then

$$(x, y)^* = 0 \qquad (x \in \mathscr{D}(A)).$$

From (2.2) and the fact that $JA^*Jx = Ax$ for $x \in \mathscr{D}(A)$, this is the same as

$$(y, x) + (JA^*Jy, Ax) = 0 \qquad (x \in \mathscr{D}(A))$$

i.e.,

$$(JA^*Jy, Ax) = (-y, x) \qquad (x \in \mathscr{D}(A)),$$

By the definition of an adjoint operator, this implies

$$JA^*Jy \in \mathscr{D}(A^*), \qquad A^*(JA^*Jy) = -y,$$

i.e.,

$$y \in \mathscr{D}(A^*JA^*J), \qquad A^*JA^*Jy = -y.$$

Since each of the above steps may be reversed, this proves

$$\mathscr{D}(JA^*J) \ominus \mathscr{D}(A) = \{y \in \mathscr{D}(A^*JA^*J) : A^*JA^*Jy = -y\}$$

which completes the proof. ▮

As an immediate consequence of Theorem 3.4 we have

COROLLARY 3.5. *If A is a closed J-symmetric operator, then the generalized defect number of A is precisely one half of the number of linearly independent solutions of $A^*JA^*Jy = -y$.*

In comparing our generalized defect number to other, possibly similar, notions we note the following facts, in which we use the definitions of kernel index, deficiency index and index, given for a general closed linear operator in [6, p. 101] and [7, p. 229–231]. In the example we give in Section 5, the generalized defect number is 1, the kernel index is 0, the deficiency index is ∞, and the index is $-\infty$. Thus our defect number does not coincide with any of these notions. Even if the deficiency index is defined to be the codimension of the *closure* of the range of the appropriate operator in the whole Hilbert space $\mathscr{H}$, they still disagree. It therefore seems that when the regularity field is empty Definition 3.2 gives a new concept. We also observe in passing that our operators are not normally solvable, semi-Fredholm or Fredholm.

4. Application to Linear Differential Operators

We now consider formal quasi-differential expressions over the interval $(a, b) \subseteq \mathbb{R}$. We assume throughout that the functions

$$p_0^{-1}, p_1, ..., p_n \tag{4.1}$$

are complex-valued, measurable over (a, b) and Lebesgue integrable on all compact subsets of (a, b). As in [15] (see also [3, 10, 25]), we define the *quasi-derivatives of a function* y, $y^{[k]}$, $0 \leqslant k \leqslant 2n$, by

$$\begin{aligned}
y^{[k]} &= \frac{d^k y}{dx^k} \qquad \text{for } 0 \leqslant k \leqslant n-1, \\
y^{[n]} &= p_0 \frac{d^n y}{dx^n}, \\
y^{[n+k]} &= p_k \frac{d^{n-k} y}{dx^{n-k}} - \frac{d}{dx}(y^{[n+k-1]}) \qquad \text{for } 1 \leqslant k \leqslant n.
\end{aligned} \tag{4.2}$$

We then define the quasi-differential expression τ by

$$\tau y = y^{[2n]}. \tag{4.3}$$

In general, the definition of τ cannot be simplified, but if the functions (4.1) are sufficiently smooth it may be written as the differential expression,

$$\tau(y) = \sum_{i=0}^{n} (-1)^{n-i}(p_i(x)\, y^{(n-i)})^{(n-i)}. \tag{4.4}$$

The endpoint a is said to be regular if $a > -\infty$ and each of the functions in (4.1) is integrable in every interval $[a, \beta]$, $\beta < b$; otherwise a is said to be singular. Similar definitions apply to b. The expression τ is said to be *regular* if it is regular at both endpoints, and otherwise is said to be *singular*. For a general discussion of quasi-differential expressions including that given by (4.2), (4.3), see [3, 25].

We are concerned with differential operators arising from τ in the Hilbert space $L^2(a, b)$. First we define the *maximal operator*, $T_{\max}$ by

$$\mathscr{D}(T_{\max}) = \{ y \in L^2(a, b) : y^{[k]} \text{ is locally absolutely continuous,} \\ 0 \leqslant k \leqslant 2n-1,\ \tau(y) \in L^2(a, b)\}, \tag{4.5}$$

$$T_{\max} y = \tau(y) \qquad (y \in \mathscr{D}(T_{\max})).$$

We then define T_0' to be the restriction of $T_{\max}$ to $\mathscr{D}(T_0')$, the set of all functions in $\mathscr{D}(T_{\max})$ with support contained in a compact subinterval of (a, b). By [25, Lemma 4], T_0' is densely defined and closable in $L^2(a, b)$. Thus we may define the *minimal operator* corresponding to τ, T_0, to be the closure of T_0' in $L^2(a, b)$. For a discussion of the comparison between the cases when the functions (4.1) are real-valued and complex-valued, respectively, see [10, 17, 26]. In the complex-valued case, τ is not formally symmetric but is formally J-symmetric if we take J to be the usual operation of complex conjugation of functions in $L^2(a, b)$. Instead of seeking boundary conditions which yield selfadjoint operators, one looks for those which yield J-selfadjoint operators.

Before proceeding we need some preliminary results which may be deduced from more general results in [25] (see also [3, 17]). The proofs are analogous to the corresponding proofs for real-valued coefficients given in [15, Sect. 17, 18].

LEMMA 4.1 (Lagrange's identity).

$$\tau(y)\, z - y\tau(z) = \frac{d}{dx}\, [y, \bar{z}] \tag{4.6}$$

for any y, z in $\mathscr{D}(T_{\max})$, where

$$[y, z] = \sum_{k=1}^{n} \{ y^{[k-1]} \bar{z}^{[2n-k]} - y^{[2n-k]} \bar{z}^{[k-1]} \}. \tag{4.7}$$

LEMMA 4.2. (i) *If J denotes the (usual) operation of complex conjugation in $L^2(a, b)$, then T_0 is a closed, J-symmetric operator and $T_{\max} = JT_0^* J$.*

(ii) *For any y and z in $\mathscr{D}(T_{\max})$ the limits $[y, \bar{z}]_a = \lim_{x \to a^+} [y, \bar{z}]$ and $[y, \bar{z}]_b = \lim_{x \to b^-} [y, \bar{z}]$ exist and we have*

$$\int_a^b \tau(y)\, z = [y, \bar{z}]_b - [y, \bar{z}]_a + \int_a^b y\tau(z). \tag{4.8}$$

(iii) $\mathscr{D}(T_0) = \{y \in \mathscr{D}(T_{\max}) : [y, \bar{z}]_b - [y, \bar{z}]_a = 0$ *for all* z *in* $\mathscr{D}(T_{\max})\}$.

It is clear from Lemma 4.2(i) that all J-selfadjoint extensions T of T_0 satisfy $T_0 \subseteq T \subseteq T_{\max}$. More precisely,

LEMMA 4.3 [10, Lemma 4.5]. *A linear manifold $\mathscr{D}'$ in $L^2(a, b)$ is the domain of definition of a J-selfadjoint extension of T_0 if and only if $\mathscr{D}'$ satisfies the following conditions*:

(i) $\mathscr{D}(T_0) \subseteq \mathscr{D}' \subseteq \mathscr{D}(T_{\max})$,

(ii) $[y, \bar{z}]_b - [y, \bar{z}]_a = 0$ *for all* y, z *in* $\mathscr{D}'$,

(iii) *every* z *in* $\mathscr{D}(T_{\max})$ *satisfying the condition* $[y, \bar{z}]_b - [y, \bar{z}]_a = 0$ *for all* y *in* $\mathscr{D}'$, *belongs to* $\mathscr{D}'$.

Using our definition of the *generalized* defect number we have from Lemma 4.2(i) and (2.3) that

$$\operatorname{def} T_0 = \tfrac{1}{2} \dim \mathscr{D}(T_{\max}) \ominus \mathscr{D}(T_0). \tag{4.9}$$

This coincides with the mean deficiency index for τ, as defined in [8]. It is observed in [8] that for a general differential expression, the mean deficiency index need not be an integer. For any formally J-symmetric quasi-differential expression of the form (4.3), Theorem 3.1 may be interpreted as saying that the mean deficiency index is indeed a nonnegative integer (for we shall see shortly that it cannot be infinite). Applying Theorem 3.4 to T_0 gives

THEOREM 4.4.

$$\mathscr{D}(T_{\max}) = \mathscr{D}(T_0) \oplus \{y \in \mathscr{D}(JT_{\max}JT_{\max}) : JT_{\max}JT_{\max}y = -y\}.$$

In the case when the coefficients of τ are real-valued, $JT_{\max} = T_{\max}J$ so $JT_{\max}JT_{\max}y = -y$ becomes $T^2_{\max}y = -y$ and the above decomposition is equivalent to the familiar

$$\mathscr{D}(T_{\max}) = \mathscr{D}(T_0) \oplus \{y \in \mathscr{D}(T_{\max}) : \tau y = iy\} \oplus \{y \in \mathscr{D}(T_{\max}) : T_{\max}y = -iy\}.$$

Here, the generalized defect number coincides with the (equal) deficiency indices for the now formally symmetric expression τ.

Returning to the general case, we use $\bar{\tau}$ to denote the formal adjoint of τ, i.e.,

$$\bar{\tau} = J\tau J.$$

Since $\bar{\tau}$ and τ are both quasi-differential expressions of the type considered by Zettl in [25], we may use the theory of [2] concerning products of such expressions to define the product, or composition, $\bar{\tau}\tau(y)$, as a quasi-differential expression, which we denote by $l(\cdot)$, thus

$$l(y) = \bar{\tau}\tau y. \tag{4.10}$$

We may now deduce

COROLLARY 4.5. (i) Def T_0 *is equal to half the number of linearly independent solutions of the equation* $\bar{\tau}\tau(y) = -y$ *for which both* y *and* $\tau(y)$ *are in* $L^2(a, b)$.

(ii) $0 \leqslant \operatorname{def} T_0 \leqslant 2n$

(iii) *If* τ *is regular at* a, *then* $n \leqslant \operatorname{def} T_0 \leqslant 2n$.

Proof. Part (i) is an immediate consequence of (4.5), (4.9), and Theorem 4.4. Part (ii) then follows from the fact that $\bar{\tau}\tau(y) = -y$ is a linear quasi-differential equation of $4n$th order. Finally, part (iii) then follows from the fact that if a is regular, one can construct $2n$ functions in $\mathscr{D}(T_{\max})$ which are linearly independent modulo $\mathscr{D}(T_0)$, as in the real case (see [3]). ▌

Remark 1. It now follows that $\operatorname{def} T_0$ is finite.

Remark 2. Since the operator $T_0^* T_0 = JT_{\max} JT_0$ is selfadjoint by [7, Theorem 3.24, p. 275], it is not difficult to see that the minimal operator determined by the quasi-differential expression $\bar{\tau}\tau$ (see [25]) must be symmetric.

Remark 3. Parts (ii) and (iii) were proved in [26] when $\Pi(T_0)$ is not empty.

In view of Remark 1, we may apply Theorem 3.3 to simplify Lemma 4.3 as follows.

THEOREM 4.6. *A linear manifold* $\mathscr{D}'$ *in* $L^2(a, b)$ *is the domain of definition of a J-selfadjoint extension of* T_0 *if and only if* $\mathscr{D}'$ *satisfies the following conditions*:

(i) $\mathscr{D}(T_0) \subseteq \mathscr{D}' \subseteq \mathscr{D}(T_{\max})$,

(ii) $[y, \bar{z}]_b - [y, \bar{z}]_a = 0$ *for all* y, z *in* $\mathscr{D}'$,

(iii) $\dim \mathscr{D}' \ominus \mathscr{D}(T_0) = \operatorname{def} T_0$.

We are now in a position to state the main result characterizing the boundary conditions which determine all the J-selfadjoint extensions of T_0.

THEOREM 4.7 (cf. [10, Theorem 4.6]). *Let* $m = \operatorname{def} T_0$. *For arbitrary*

$w_1, \ldots, w_m$ belonging to $\mathscr{D}(T_{\max})$ which are linearly independent modulo $\mathscr{D}(T_0)$ and which satisfy the relations

$$[w_j, \bar{w}_k]_b - [w_j, \bar{w}_k]_a = 0 \qquad (j, k = 1, \ldots, m) \tag{4.11}$$

the set of all functions y in $\mathscr{D}(T_{\max})$ which satisfy the conditions

$$[y, \bar{w}_k]_b - [y, \bar{w}_k]_a = 0 \qquad (k = 1, \ldots, m) \tag{4.12}$$

is the domain of a J-selfadjoint extension of T_0. Conversely, all J-selfadjoint extensions of T_0 are of this form.

Proof. In view of Lemma 4.2 (iii) and the fact that $[y, \bar{y}] \equiv 0$ for any y in $\mathscr{D}(T_{\max})$, conditions (i)–(iii) of Theorem 4.6 are equivalent to $\mathscr{D}'$ being of the form

$$\mathscr{D}' = \mathscr{D}(T_0) \dotplus L(w_1, \ldots, w_m),$$

where $w_1, \ldots, w_m$ lie in $\mathscr{D}(T_{\max})$, are linearly independent modulo $\mathscr{D}(T_0)$ and satisfy (4.11). The formulation of $\mathscr{D}'$ as being the set of all y in $\mathscr{D}(T_{\max})$ satisfying (4.12) is then immediate. ▮

Remark 1. We recall that if T is a J-selfadjoint extension of T_0, then $T \subseteq T_{\max}$ so $Ty = \tau(y)$ for any $y \in \mathscr{D}(T)$.

Remark 2. The example which is given in Section 5 demonstrates that Theorem 4.7 is a proper extension of [10, Theorem 4.6].

Remark 3. The proofs given here for Theorem 4.7 and its prerequisite results, are shorter and simpler, even when $\Pi(T_0)$ is not empty than those employed by Knowles in [10]. The prerequisites here are the results of Galindo [4], Knowles [9], and Section 3 of the present work.

In the regular case, it is known that $\Pi(T_0)$ covers the whole complex plane. Thus whilst Theorem 4.7 is applicable, it yields nothing new and the details are contained in [10, Theorem 4.7] which can be deduced from the above result. However, the application of Theorem 4.7 to the case when a is regular but b is not, is of more interest. We need one additional result first, namely,

LEMMA 4.8 ([10, Lemma 4.8], cf. [15, Sect. 18.3], [8, p. 19]). *If τ is regular at a and singular at b, then* def $T_0 = n$ *if and only if for arbitrary y and z in $\mathscr{D}(T_{\max})$,*

$$[y, \bar{z}]_b = 0. \tag{4.13}$$

A major consequence of this is that the terms $[w_j, \bar{w}_k]_b$ and $[y, \bar{w}_k]_b$ in (4.11) and (4.12) then disappear. Thus, as in [10], Theorem 4.7 simplifies to

THEOREM 4.9 (cf. [10, Theorem 4.9]). *Let τ be regular at a and singular at b, and assume* def $T_0 = n$. *Then the linear manifold in $\mathscr{D}(T_{\max})$ determined by linearly independent boundary conditions at a of the form*

$$\sum_{k=1}^{2n} \alpha_{jk} y^{[k-1]}(a) = 0, \qquad j = 1,..., n \tag{4.14}$$

with

$$\sum_{\nu=1}^{n} \alpha_{j\nu}\alpha_{k,2n-\nu+1} - \sum_{\nu=1}^{n} \alpha_{j,2n-\nu+1}\alpha_{k\nu} = 0, \qquad j, k = 1,..., n \tag{4.15}$$

is the domain of a J-selfadjoint extension of T_0. Conversely, every J-selfadjoint extension of T_0 is of this form.

If we set $n = 1$ in (4.1)–(4.3) when a is regular, τ becomes

$$\tau(y) = -(p_0(x)\, y')' + p_1(x)\, y, \qquad a \leqslant x < b \tag{4.16}$$

and condition (4.15) is trivially satisfied. Thus Theorem 4.9 becomes

COROLLARY 4.10 (cf. [10, Corollary 4.10]). *If T_0 is the minimal operator corresponding to the formal operator τ defined by* (4.16), *and* def $T_0 = 1$, *then the J-selfadjoint extensions T_γ of T_0 are precisely given by*

$$\mathscr{D}(T_\gamma) = \{ y \in \mathscr{D}(T_{\max}) \colon \gamma_1 y(a) + \gamma_2 p_0(a)\, y'(a) = 0 \},$$
$$T_\gamma y = \tau(y) \qquad (y \in \mathscr{D}(T_\gamma)),$$

where $\gamma = (\gamma_1, \gamma_2)$ is an arbitrary nonzero number in $\mathbb{C}^2$.

5. FURTHER CONSEQUENCES

In this final section, we consider a few consequences of the results proved in Section 4. First, we wish to look specifically at the question of what does happen when $\Pi(T_0)$ is empty. In particular, whether this would imply anything about the value of def T_0, and hence about the nature of the J-selfadjoint extensions of T_0. We shall concentrate on the situation when a is a regular endpoint for τ, and hence τ may be considered on the interval $[a, b)$. We need two preliminary results.

LEMMA 5.1 (see [16]). *If, for some $\lambda_0 \in \mathbb{C}$, there are $2n$ linearly independent solutions of $\tau(y) = \lambda_0 y$ in $L^2[a, b)$, then $\lambda_0 \in \Pi(T_0)$ and indeed $\Pi(T_0) = \mathbb{C}$.*

To evaluate def T_0, we need to use Corollary 4.5. The matrix which determines the quasi-differential expression $l(\cdot)$ in (4.10) is given by [2, Theorem 1], from which it is clear that not only is the corresponding minimal operator symmetric (as observed earlier) but also, this matrix satisfies the stronger symmetry condition, (3.3) of [3]. This enables us to apply [3, Theorem 9.1] to obtain

LEMMA 5.2. *Let $l(\cdot)$ be defined by* (4.10). *If all the solutions of $l(y) = \lambda_0 y$ are in $L^2[a, b)$ for some λ_0 in $\mathbb{C}$, then all solutions of $l(y) = \lambda y$ are in $L^2[a, b)$ for any λ in $\mathbb{C}$.*

We may now prove

THEOREM 5.3. *Let T_0 be the minimal operator associated with τ as defined by* (4.3) *on the interval $[a, b)$. If $\Pi(T_0)$ is empty, then* def $T_0 \neq 2n$. *In particular if $\Pi(T_0)$ is empty and $n = 1$, then* def $T_0 = 1$.

Proof. If def $T_0 = 2n$, then from Corollary 4.5 (i), $l(y) = -y$ has all its solutions in $L^2[a, b)$. Setting $\lambda_0 = -1$ and $\lambda = 0$ in Lemma 5.2, we may deduce that all solutions of $l(y) = 0$ are in $L^2[a, b)$. It is then clear from (4.10) that all solutions of $\tau(y) = 0$ are in $L^2[a, b)$ as they form some of the solutions of $l(y) = 0$. But then Lemma 5.1 implies $\Pi(T_0) = \mathbb{C}$. Thus if $\Pi(T_0)$ is empty, we cannot have def $\mathrm{T}_0 = 2n$. In particular, if $n = 1$, Corollary 4.5(iii) gives that def T_0 is either 1 or 2, so if $\Pi(T_0)$ is empty we must have def $T_0 = 1$. ∎

Remark 1. If $n > 1$ it remains an open question as to what values of def T_0 are possible when $\Pi(T_0)$ is empty. It seems likely that either all values between n and $2n - 1$ (inclusive) are possible, or else def T_0 is necessarily n whenever $\Pi(T_0)$ is empty. We would conjecture that the former is true.

Remark 2. It is also an open question as to how many of the solutions of $\tau(y) = \lambda y$ may be in $L^2[a, b)$ for any λ in $\mathbb{C}$ when $\Pi(T_0)$ is empty, except that we know from above that not all of them are in $L^2[a, b)$.

For the remainder of this section, we concentrate on the case $n = 1$, when τ is given by (4.16). If the coefficients p_0 and p_1 in (4.16) are real-valued, it is well-known that the whole of $\mathbb{C} \backslash \mathbb{R}$ is contained in $\Pi(T_0)$ and τ may be classified as being of limit-point or limit-circle type, according to the nature of p_0 and p_1. This classification, which was first described by Weyl [24] now has a natural analogue when p_0 and p_1 are permitted to be complex-valued.

THEOREM 5.4. *Let T_0 be the minimal operator corresponding to the formal operator τ defined by* (4.16), *then*

(a) *the following are equivalent*:

(i) def $T_0 = 1$;

(ii) *precisely one boundary condition is needed to determine the J-selfadjoint extensions of* T_0, *which are then given by Corollary* 4.10;

(iii) *there is at most one solution of* $\tau(y) = \lambda y$ *in* $L^2[a, b)$ *for any (and hence every)* λ *in* $\mathbb{C}$;

(b) *the following are equivalent*:

(i) def $T_0 = 2$;

(ii) *precisely two boundary conditions are needed to determine the J-selfadjoint extensions of* T_0;

(iii) *there are two linearly independent solutions of* $\tau(y) = \lambda y$ *in* $L^2[a, b)$ *for any (and hence every)* λ *in* $\mathbb{C}$;

(c) *for any given* p_0 *and* p_1, *the conditions of either* (a) *or* (b) *are satisfied.*

Proof. That (a)(i) and (a)(ii) are equivalent follows from Theorem 4.7, Lemma 4.8, and Corollary 4.10. Similarly, (b)(i) and (b)(ii) are equivalent. It is then clear from Corollary 4.5 that def T_0 is either 1 or 2 so (a)(i) and (b)(i) are mutually exclusive and cover all possibilities. If (b)(iii) holds, then from Lemma 5.1 and Definition 2.3, def $T_0 = 2$. On the other hand, if def $T_0 = 2$, then it follows from the proof of Theorem 5.3 that all solutions of $\tau(y) = 0$ are in $L^2[a, b)$ and hence by Lemma 5.1, (b)(iii) holds. It now follows that (a)(iii) is equivalent to (a)(i), which completes the proof. ∎

As in the case of real-valued coefficients we may refer to (a) above as the *limit-point* alternative and (b) as the *limit-circle* alternative. The above classification was described in [11, Theorem 2.4] under the assumption that $\Pi(T_0)$ was not empty. Furthermore Theorem 5.3 tells us the following.

COROLLARY 5.5. *If* τ *is defined by* (4.16) *and* $\Pi(T_0)$ *is empty, then* τ *is in the limit-point case, i.e.,* def $T_0 = 1$ *and the J-selfadjoint extensions of* T_0 *are the operators* T_γ *defined in Corollary* 4.10.

In [11, Sect. 3], some very general criteria were derived for the equation $\tau(y) = 0$ not to have all its solutions in $L^2[a, \infty)$. Theorem 5.4 proves that these results, together with those in [17] referred to there, may now be regarded as being limit-point criteria with no assumption being needed concerning $\Pi(T_0)$ (see also the results of Read [22] in this direction).

It is, however, ironical that the geometry of the $m(\lambda)$-functions which gave the limit-point, limit-circle dichotomy its name, is the one major part of the theory which has not been fully extended to the complex case. For work in this direction, see [19, 23] (cf. [14]).

We conclude with an example which demonstrates the possibility that $\Pi(T_0)$ may be empty and which therefore illustrates some of the foregoing results.

EXAMPLE 5.6. In [14] McLeod considered the following example in which $n=1$.

$$\tau(y(x)) = -y''(x) - 2ie^{2(1+i)x}y(x) \qquad \text{on } [0, \infty),$$

where $i=\sqrt{-1}$. He showed that no solution of $\tau(y)=\lambda y$ is in $L^2[0, \infty)$ for any λ in $\mathbb{C}$. This implies (see [11, Sect. 2; 26]) that $\Pi(T_0)$ is empty (and, in fact, that the essential spectrum of T_0 covers the whole complex plane). Using Definition 3.2 for the (generalized) defect number we may conclude from Corollary 5.5 that def $T_0=1$, i.e., that $\mathscr{D}(T_{\max})$ is a two-dimensional extension of $\mathscr{D}(T_0)$; also, the J-selfadjoint extensions of T_0 are the operators T_γ, obtained by placing a single boundary condition on at the endpoint 0; furthermore, Lemma 4.8 gives that $[y, \bar{z}]_\infty = 0$ for all y, z in $\mathscr{D}(T_{\max})$. These properties had not previously been obtained and it would seem that the operator theoretic results of Section 3 above, are needed in order to derive them.

Final Remark. We observe that although we have considered quasi-differential operators in $L^2(a, b)$ in sections 4 and 5, the corresponding results remain true in the weighted space $L^2_w(a, b)$, if the weight function w is positive almost everywhere, and locally integrable in (a, b).

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